Quantitative Literacy:
Thinking Between the Lines

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Chapter 8:
Voting and Social Choice

Chapter 8: Voting and Social Choice
Lesson Plan

- Measuring voting power: Does my vote count?
- Voting systems: How do we choose a winner?
- Fair division: What is a fair share?
- Apportionment: Am I represented?

Learning Objectives:

- Understand the mathematically sound ways of fair division of assets in situations such as divorce or inheritance.
  - Divide and choose procedure
  - Adjusted winner procedure
  - The Knaster procedure

Chapter 8 Voting and Social Change
8.3 Fair Division: What is a fair share?

- **Divide-and-Choose Procedure:** One person divides the items into two parts and the other person chooses which part he or she wants.

- The *Lone-Divider Method* applies this procedure to parties of 3:

  Person 1 divides the assets into three parts, persons 2 and 3 then choose between the piles.

  If they choose differently person 1 gets the remaining pile.

  If they choose the same, person 1 chooses which pile they want and persons 2 and 3 mix the remaining piles and perform the divide-and-choose procedure.

8.3 Fair Division: What is a fair share?

- **Adjusted winner procedure:** Two people assign points to bid on each item, assigning a total of 100 points.

  Initial division of the items gives each item to the person offering the highest bid.

  The division is then adjusted based on the ratio of the bids for each item so that ultimately each person receives a group of items whose bid totals are the same for each person.

- **Example:** Suppose my sister and I want to divide an inheritance. The assets consist of a guitar, a jewelry collection, a car, a small library, and a certain amount of cash. To start the procedure, each of us takes 100 points and divides those points among the assets. In a division of assets, point values of two siblings follow:

8.3 Fair Division: What is a fair share?

- **Example (cont.):**

My Points	Item	Sister's Points
35	Guitar	10
10	Jewelry	10
20	Car	40
15	Library	10
20	Cash	30

- Initial round: I get the guitar and the library for a total of 50 points, the sister gets the car and the cash for a total of 70 points.

  The tied item, the jewelry, goes to point leader, my sister.

  So she is 80% satisfied while I am only 50% satisfied. The points must be adjusted.

- To make the values even, my sister must share some of her property.

  Each item my sister won can be calculated as: $\frac{\text{Sister's Bid}}{\text{My Bid}}$

- **Example (cont.):**

  Note my sister's bid goes on top because she is the point leader.

  Arrange the ratios in increasing order:

$$\text{Ratio for jewelry} = \frac{10}{10} = 1$$

$$\text{Ratio for cash} = \frac{30}{20} = 1.5$$

$$\text{Ratio for car} = \frac{40}{20} = 2.0$$

  Items are then transferred until an item changes the point leader

  First the jewelry is transferred to me: my expended points is 60 and my sister's is 70.

  The next item, the cash, is the **Critical Item**; it changes the point leader. Just enough of the critical item is transferred to make the score come out even.

8.3 Fair Division: What is a fair share?

- **Example (cont.):**
- The equation for dividing the critical item is as follows:

$$\text{My score} + p \text{ percent of my cash bid} = \text{Sister's score} - p \text{ percent of her cash bid}$$

- This gives the equation: $60 + 20p = 70 - 30p$ or $p = 0.20$.

So I get the guitar, the library, and 20% of the cash.

My sister gets the car and the remaining 80% of the cash.

The value scores are:

My total points: $35 + 10 + 15 + 20 \times 0.20 = 64$

My sister's total points: $40 + 30 \times 0.80 = 64$

- Both are 64% satisfied.

Chapter 8 Voting and Social Change
8.3 Fair Division: What is a fair share?

- **The Adjusted Winner Procedure:** Each of 2 people makes a bid totaling 100 points on a list of items to be divided.
- **Step 1: Initial division of items:** Each item goes to the higher bidder. Tied items are held for now. The higher score is the point leader, the lesser score is the trailer.
- **Step 2: Tied items:** Tied items go to the leader.
- **Step 3: Calculate leader/trailer ratios:** For each item belonging to the leader, calculate the ratio: $\frac{\text{leader's bid}}{\text{trailer's bid}}$
- **Step 4: Transference of some items from leader to trailer:** Transfer items from leader to trailer in order of increasing ratios as doing so does not change the lead. The item to change the lead is the *critical item*.
- **Step 5: Division of the critical item:** Give p percent of the critical item to the trailer from leader with the following equation:

$$\text{Trailer's score} + p \times \text{Trailer's bid} = \text{Leader's score} - p \times \text{Leader's bid}$$

8.3 Fair Division: What is a fair share?

- **Example:** Divide the following items from an inheritance:

John	Item	Faye
50	Vacation condominium	65
20	Red 1962 GT Hawk	15
20	Family silver set	15
10	Dad's Yale cap/gown	5

- **Solution:** Initially John gets the Hawk, the silver, and the cap and gown. Faye gets the condominium.

  Faye is the leader; her condo is the critical item so is considered for the transfer.

- **Solution (cont.):** John's score $+ p \times$ John's bid $=$ Faye's score $- p \times$ Faye's bid

$$50 + 50p = 65 - 65p$$

$$p = \frac{15}{115} = 0.13$$

John takes 13% of the condo from Faye.

Faye gets 87% ownership of the condo.

John gets the Hawk, the silver, the cap/gown, and 13% ownership of the condo.

The divided ownership of the condo could be satisfied by splitting use of the condo up with John getting 7 weeks and Faye getting 45 weeks each year.

8.3 Fair Division: What is a fair share?

- **Example:** Use the adjusted winner procedure to fairly divide the following items:

Anne	Item	Becky
50	Stereo	40
5	CDs	5
6	Tapes	5
31	Cabinet	10
8	TV	40

- **Solution:** Initially Anne gets the stereo, the tapes, and the cabinet for 87 points. Becky gets the TV for a point total of 40.

Anne is the leader so she gets the tied item, the CDs, bringing her points to 92.

Anne's items are compared against Becky's:

$$\text{Ratio for CDs} = \frac{5}{5} = 1$$

$$\text{Ratio for tapes} = \frac{6}{5} = 1.2$$

$$\text{Ratio for stereo} = \frac{50}{40} = 1.25$$

$$\text{Ratio for cabinet} = \frac{31}{10} = 3.1$$

- **Solution (cont.):** The CDs and tapes are transferred. Anne's score is now 81 and Becky's is 50. The stereo is the critical item.

  $\text{Becky's score} + p \times \text{Becky's bid} = \text{Anne's score} - p \times \text{Anne's bid}$

  $$p = \frac{31}{90} = 0.34$$

  Becky gets the CDs, the tapes, the TV, and 34% of the stereo.

  Anne gets the cabinet and the remaining 66% of the stereo.

  This is as fair as it can be for a party of two, considering of course that neither party changes their mind about point allocation.

  There are some problems with this procedure, for example, parties of three or more, critical items being non-dividable, etc.

8.3 Fair Division: What is a fair share?

- The *Knaster procedure* is a method for dividing items among several parties, and it does not require dividing ownership of items. The procedure is based on having the parties assign monetary value to each item, their *bid*. The bidding is without knowledge of others' bids.

- **Example:** Assume there is one item to be divided, an automobile. The four people involved are each entitled to $1/4$ of the car. Each person places a bid and each will end up with a value of at least $1/4$ of his bid. Suppose the bids are as follows:

Person	Abe	Ben	Caleb	Dan
Bid	$30,000	$23,000	$28,000	$25,000

8.3 Fair Division: What is a fair share?

- **Example:**

Person	Abe	Ben	Caleb	Dan
Bid	$30,000	$23,000	$28,000	$25,000

The highest bidder, Abe, gets the car.

But that is the entire estate and he is only entitled to $1/4$, which is $\$7500$. So he puts the remaining $3/4$, $\$22{,}500$, into a kitty.

Each of the others withdraws $1/4$ of what they assigned as the value of the car:

Ben takes $1/4$ of $\$23{,}000 = \$5750$

Caleb withdraws $1/4$ of $\$28{,}000 = \$7000$

Dan takes $1/4$ of $\$25{,}000 = \$6250$

The total withdrawn is $5750 + 7000 + 6250 = \$19{,}000$

8.3 Fair Division: What is a fair share?

- **Example (cont.):**

  This leaves a kitty of $\$3500$ to be divided equally among the four people. Each person receives $1/4$ of the value he assigned to the car plus $\frac{\$3500}{4} = \$875$.

  The following table summarizes the transaction:

Person		**Abe**	**Ben**	**Caleb**	**Dan**	**Kitty**
Bid		**$\$30,000$**	**$\$23,000$**	**$\$28,000$**	**$\$25,000$**	
Car award		Car				0
To kitty		−22,500				22,500
From kitty			5750	7000	6250	3500
From kitty		875	875	875	875	0
Final share	Car-less	$21,625	$6625	$7825	$7125	$0

The Knaster Procedure, or method of sealed bids:

Step 1: Each person bids a secret dollar amount for each item to be divided (sealed bids).

Step 2: For each item separately:

1. The item is given to the person who bid the most.
2. The winner contributes the difference of their vote and his or her share of the item to a kitty.
3. Those who don't win the item withdraw their share of each of their bids from the kitty.

Step 3: The money left in the kitty is divided equally among the bidders.

8.3 Fair Division: What is a fair share?

- **Example:** A family farm is to be divided, parts of the farm are to be bid on by three family members.

Bid	Julie	Anne	Steve
Farmhouse	$120,000	$60,000	$75,000
Wheat fields	210,000	450,000	390,000
Cattle operations	300,000	750,000	600,000

Determine the fair division of the farm using the Knaster procedure.

- **Solution:** Each part of the farm is given to the highest bidder.

Julie gets the farmhouse.

Anne gets the wheat field and the cattle operations.

Steve will not get any of the farm, but he will get cash.

The divisions of each piece of the farm and its cash value follow.

- **Solution (cont.):**

Farmhouse: Julie gets the farmhouse. She bid $\$120{,}000$ but is entitled to $1/3$ of that, $\$40{,}000$, so she puts $\$80{,}000$ in the kitty. Anne and Steve are entitled to $1/3$ of their respective bids.

- Anne takes $1/3$ of $\$60{,}000 = \$20{,}000$.
- Steve takes $1/3$ of $\$75{,}000 = \$25{,}000$.

The total removed from kitty is $20{,}000 + 25{,}000 = \$45{,}000$.

Bid	Julie	Anne	Steve	Kitty
Farmhouse	$\$120{,}000$	$\$60{,}000$	$\$75{,}000$	
Farmhouse award	Farmhouse			$\$0$
To kitty	$-\$80{,}000$			$\$80{,}000$
From kitty		$\$20{,}000$	$\$25{,}000$	$\$35{,}000$

This leaves $\$80{,}000 - \$45{,}000 = \$35{,}000$ in the kitty.

- **Solution (cont.):**

Wheat fields: Anne gets the wheat fields. She bid $\$450{,}000$ but is entitled to $1/3$ of that, $\$150{,}000$, so she puts $\$300{,}000$ in the kitty. Julie and Steve are entitled to $1/3$ of their respective bids.

- Julie takes $1/3$ of $\$210{,}000 = \$70{,}000$.
- Steve takes $1/3$ of $\$390{,}000 = \$130{,}000$.

The total removed from kitty is $70{,}000 + 130{,}000 = \$200{,}000$.

Bid	Julie	Anne	Steve	Kitty
Wheat fields	$\$210{,}000$	$\$450{,}000$	$\$390{,}000$	
Wheat fields award		Wheat fields		$\$35{,}000$
To kitty		$-\$300{,}000$		$\$335{,}000$
From kitty	$\$70{,}000$		$\$130{,}000$	$\$135{,}000$

- **Solution (cont.):**

Cattle ops.: Anne gets the cattle operations. She bid $\$750{,}000$ but is entitled to $1/3$ of that, $\$250{,}000$, so she puts $\$500{,}000$ in the kitty. Julie and Steve are entitled to $1/3$ of their respective bids.

- Julie takes $1/3$ of $\$300{,}000 = \$100{,}000$
- Steve takes $1/3$ of $\$600{,}000 = \$200{,}000$

The total removed from kitty is $100{,}000 + 200{,}000 = \$300{,}000$

Bid	Julie	Anne	Steve	Kitty
Cattle operations	$\$300{,}000$	$\$750{,}000$	$\$600{,}000$	
Cattle ops. award		Cattle ops.		$\$135{,}000$
To kitty		$-\$500{,}000$		$\$635{,}000$
From kitty	$\$100{,}000$		$\$200{,}000$	$\$335{,}000$
Kitty distribution	$\$111{,}667$	$\$111{,}667$	$\$111{,}667$	

- **Solution (cont.):**

After the item distribution, there is $\$335{,}000$ left in the kitty to be divided among the three.

$$335{,}000/3 = \$111{,}667 \text{ each}$$

- Julie gets the Farmhouse and cash totaling:

$$-\$80{,}000 + 70{,}000 + 100{,}000 + 111{,}667 = \$201{,}667$$

- Anne gets the wheat fields, the cattle operations and cash totaling:

$$\$20{,}000 - 300{,}000 - 500{,}000 + 111{,}667 = -\$668{,}333$$

That is, Anne needs to pay out that much.

- Steve gets no property but cash totaling:

$$\$25{,}000 + 130{,}000 + 200{,}000 + 111{,}667 = \$446{,}667$$

Chapter 8 Voting and Social Change: **Chapter Summary**

- **Voting power:** does your vote count
  - coalitions, quotas, swing voters
  - Banzhaf power index
  - Stanley-Shubik power index

- **Voting systems:** choosing a winner
  - plurality
  - top-two runoff
  - elimination runoff
  - Borda count
  - Condorcet winner

Chapter 8 Voting and Social Change: **Chapter Summary**

- **Fair division:**
  - Methods: divide and choose
    adjusted winner procedure

- **Apportionment:**
  - Hamilton's method
  - Jefferson's method
  - Adjusted divisor methods
  - Huntington-Hill method